Solvable Quintic Equations with Commensurable Coefficients.

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OBJECT OF THE PAPER.

§1. Some time ago, in the *American Journal of Mathematics* (Vol. VI, page 103), the present writer sketched a general method for finding the roots of solvable irreducible equations of the fifth degree. The method was partially developed, and its application to certain forms of quintic equations was shown. It is now proposed to give the method the farther development necessary to make it applicable, by a definite and certain process, and without any difficulty beyond the labor of operation, to all solvable irreducible quintics having commensurable coefficients. The following equations will be solved as examples of the application of the theory:

1. $x^5 + 3x^2 + 2x - 1 = 0.$
2. $x^5 - 10x^3 - 20x^2 - 1505x - 7412 = 0.$
3. $x^5 + \frac{625}{4}x + 3750 = 0.$
4. $x^5 - \frac{22}{5}x^3 - \frac{11}{25}x^2 + \frac{11 \times 42}{125}x + \frac{11 \times 89}{3125} = 0.$
5. $x^5 + 20x^3 + 20x^2 + 30x + 10 = 0.$
6. $x^5 + 320x^2 - 1000x + 4288 = 0.$
7. $\left(\frac{x}{10}\right)^5 + 40\left(\frac{x}{10}\right)^2 - 69\left(\frac{x}{10}\right) + 108 = 0.$
8. $x^5 - 20x^3 + 250x - 400 = 0.$
9. $x^5 - 5x^3 + \frac{85}{8}x - \frac{13}{2} = 0.$
10. $x^5 + \frac{20x}{17} + \frac{21}{17} = 0.$

11. $x^5 - \frac{4x}{13} + \frac{29}{65} = 0.$

12. $x^5 + \frac{10x}{13} + \frac{3}{13} = 0.$

13. $x^5 + 110(5x^3 + 60x^2 + 800x + 8320) = 0.$

14. $x^5 - 20x^3 - 80x^2 - 150x - 656 = 0.$

15. $x^5 - 40x^3 + 160x^2 + 1000x - 5888 = 0.$

16. $\left(\frac{x}{2}\right)^5 - 50\left(\frac{x}{2}\right)^3 - 600\left(\frac{x}{2}\right)^2 - 2000\left(\frac{x}{2}\right) - 11200 = 0.$

17. $x^5 + 110(5x^3 + 20x^2 - 360x + 800) = 0.$

18. $x^5 - 20x^3 + 320x^2 + 540x + 6368 = 0.$

19. $x^5 - 20x^3 - 160x^2 - 420x - 8928 = 0.$

20. $x^5 - 20x^3 + 170x + 208 = 0.$

The first equation in this group was brought under the notice of the writer by a mathematical correspondent; the fourth has been treated by Lagrange; the others were formed by the writer with a view to the full illustration of his theory.

THE METHOD.

§2. In the article of the *Journal* above referred to, certain principles were assumed, as having been previously established, or as being known to mathematicians. It was taken for granted that the root x of the solvable irreducible quintic

$$x^5 + p_2x^3 + p_3x^2 + p_4x + p_5 = 0 \tag{1}$$

is of the form

$$\frac{1}{5}(\Delta_1^{\frac{1}{5}} + \Delta_2^{\frac{1}{5}} + \Delta_3^{\frac{1}{5}} + \Delta_4^{\frac{1}{5}}),$$

or, putting u_1 for $\frac{1}{5}\Delta_1^{\frac{1}{5}}$, u_2 for $\frac{1}{5}\Delta_2^{\frac{1}{5}}$, u_3 for $\frac{1}{5}\Delta_3^{\frac{1}{5}}$, and u_4 for $\frac{1}{5}\Delta_4^{\frac{1}{5}}$,

$$u_1 + u_2 + u_3 + u_4,$$

where u_1, u_2, u_3 and u_4 are the roots of a quartic equation, which, when irreducible, as it is in the most general case, that includes all the others, is a uni-serial Abelian. The expressions u_1, u_2, u_3 and u_4 are such that

$$\left.\begin{aligned} u_1u_4 &= g + a\sqrt{z} \\ u_2u_3 &= g - a\sqrt{z} \end{aligned}\right\}; \tag{2}$$

and

$$\left.\begin{aligned} u_1^2u_3 &= k + c\sqrt{z} + (\theta + \phi\sqrt{z})\sqrt{(hz + h\sqrt{z})} \\ u_4^2u_2 &= k + c\sqrt{z} - (\theta + \phi\sqrt{z})\sqrt{(hz + h\sqrt{z})} \\ u_2^2u_1 &= k - c\sqrt{z} + (\theta - \phi\sqrt{z})\sqrt{(hz - h\sqrt{z})} \\ u_3^2u_4 &= k - c\sqrt{z} - (\theta - \phi\sqrt{z})\sqrt{(hz - h\sqrt{z})} \end{aligned}\right\} \tag{3}$$

where g, k, a, c, h, θ and ϕ are rational; and $z = e^2 + 1$, e being rational. It is readily seen that

$$g = -\frac{p_2}{10}, \text{ and } k = -\frac{p_3}{20}.$$

Because $u_1^5 = \frac{(u_1^2u_3)^2(u_2^2u_1)}{(u_2u_3)^2}$, it follows from (2) and (3) that

$$\left.\begin{aligned} u_1^5 &= B + B'\sqrt{z} + (B'' + B'''\sqrt{z})\sqrt{(hz + h\sqrt{z})} \\ u_4^5 &= B + B'\sqrt{z} - (B'' + B'''\sqrt{z})\sqrt{(hz + h\sqrt{z})} \\ u_2^5 &= B - B'\sqrt{z} + (B'' - B'''\sqrt{z})\sqrt{(hz - h\sqrt{z})} \\ u_3^5 &= B - B'\sqrt{z} - (B'' - B'''\sqrt{z})\sqrt{(hz - h\sqrt{z})} \end{aligned}\right\} \tag{4}$$

where B, B', B'' and B''' are rational functions of a, c, e, h, θ and ϕ. In like manner, because $u_1^3u_2 = \frac{(u_1^2u_3)(u_2^2u_1)}{u_2u_3}$, we have from (2) and (3)

$$u_1^3u_2 = A + A'\sqrt{z} + (A'' + A'''\sqrt{z})\sqrt{(hz + h\sqrt{z})},$$

where A, A', A'' and A''' are rational. The value of A is

$$A = \frac{1}{g^2 - a^2z}\{g(k^2 - c^2z) + azhe(\theta^2 - \phi^2z)\}. \tag{5}$$

From these data, the six equations, involving the six unknown quantities a, c, e, h, θ and ϕ, are (see *Journal of Mathematics* as above) obtained:

$$\left.\begin{aligned} p_4 &= -20A + 5g^2 + 15a^2z \\ p_5 &= -4B + 40acz \\ B'' &= 1 \\ B''' &= 0 \\ hz(\theta^2 + \phi^2z + 2\theta\phi) &= k^2 + c^2z - g(g^2 - a^2z) \\ h(\theta^2 + \phi^2z + 2\theta\phi z) &= 2kc - a(g^2 - a^2z) \end{aligned}\right\} \tag{6}$$

Our business is to obtain u_1^5, u_2^5, u_3^5 and u_4^5 from these equations.

§3. It will be found that a^2z is the root of an equation $F(y) = 0$, whose coefficients are rational functions of p_2, p_3, p_4 and p_5, and which, when p_2 is zero, is of the sixth degree. Since a^2z is rational, it follows that, when the coefficients of the given quintic are commensurable, the equation $F(y) = 0$ has a commensurable root. Let this be found. Then a^2z is known. The formulæ from which the equation $F(y) = 0$ is obtained give us, along with a^2z, the value of $\frac{c}{a}$. The remaining elements necessary for the determination of u_1^5, u_2^5, u_3^5 and u_4^5 may then be obtained from linear equations, without finding

a, c, e, h, θ and ϕ separately. Thus a solution of the given quintic is effected. It will be pointed out how a, c, e, h, θ and ϕ can be found separately, should we desire to obtain their values.

THE PROOF.

Case in which p_2 is zero.

§4. When $p_2 = 0$, the investigation is much simplified. By beginning with this case, and presenting a full description of it, we shall be prepared for giving an exposition, less detailed, but still sufficiently minute to make the theory intelligible, of the case in which p_2 is not assumed to be zero. When p_2 is zero, equations (2) and (5) become

$$\left.\begin{aligned} u_1u_4 &= a\sqrt{z} \\ u_2u_3 &= -a\sqrt{z} \\ \text{and}\qquad A &= -\frac{he(\theta^2-\phi^2 z)}{a} \end{aligned}\right\} \tag{7}$$

Since B is the coefficient of the rational part, B' the coefficient of $\sqrt{z}$, B'' the coefficient of $\sqrt{(hz + h\sqrt{z})}$, and B''' the coëfficient of $\sqrt{z}\sqrt{(hz + h\sqrt{z})}$, in the expansion of u_1^5, their values are given by the equations

$$\left.\begin{aligned} a^2zB &= 2k(k^2-c^2z) - a^3cz^2 + 2czhe(\theta^2-\phi^2z) \\ a^2zB' &= 2c(k^2-c^2z) + a^3kz + 2khe(\theta^2-\phi^2z) \\ a^2zB'' &= 2(k^2-c^2z)\theta - \frac{2(k^2+c^2z)(\theta+\phi z)}{e} + \frac{z(\theta+\phi)(4kc+a^3z)}{e} \\ a^2zB''' &= 2(k^2-c^2z)\phi + \frac{2(k^2+c^2z)(\theta+\phi)}{e} - \frac{(\theta+\phi z)(4kc+a^3z)}{e} \end{aligned}\right\} \tag{8}$$

The equations (6), when p_2 is zero, become

$$\left.\begin{aligned} p_4 &= -20A + 15a^2z \\ p_5 &= -4B + 40acz \\ B'' &= 1 \\ B''' &= 0 \\ hz(\theta^2+\phi^2z+2\theta\phi) &= k^2+c^2z \\ h(\theta^2+\phi^2z+2\theta\phi z) &= 2kc + a^3z \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} \text{Let}\qquad y &= a^2z \\ \text{and}\qquad t &= \frac{c}{a} \end{aligned}\right\} \tag{10}$$

Substituting in the first of equations (8) the value of B obtained from the second of equations (9), and the value of $he(\theta^2 - \phi^2 z)$ obtained from (7), and making use of (10),

$$p_5 y = -8k(k^2 - t^2 y) + 44ty^2 + 8tyA.$$

Therefore, by the first of equations (9),

$$8kyt^2 + \left(50y^2 - \frac{2}{5}p_4 y\right)t = 8k^3 + p_5 y. \tag{11}$$

§5. Again, from the last two of equations (9), because $z = e^2 + 1$,

$$he^2(\theta^2 + \phi^2 z) = (k^2 + c^2 z) - (2kc + a^3 z).$$

Therefore, by (10),

$$he^2(\theta^2 + \phi^2 z) = (k^2 + t^2 y) - a(2kt + y). \tag{12}$$

Similarly, from the last two of equations (9),

$$2hze^2\theta\phi = az(2kt + y) - (k^2 + t^2 y). \tag{13}$$

And $$(\theta^2 - \phi^2 z)^2 = (\theta^2 + \phi^2 z)^2 - 4z(\theta^2\phi^2).$$

Therefore, from (12) and (13),

$$(he^2)^2(\theta^2 - \phi^2 z)^2 = \{(k^2 + t^2 y) - a(2kt + y)\}^2 - \frac{1}{z}\{az(2kt + y) - (k^2 - t^2 y)\}^2$$

$$\therefore zh^2 e^2(\theta^2 - \phi^2 z)^2 = (k^2 + t^2 y)^2 - y(2kt + y)^2.$$

Hence from the value of $he(\theta^2 - \phi^2 z)$ in (7),

$$(k^2 + t^2 y)^2 - y(2kt + y)^2 = za^2 A^2 = yA^2.$$

But, by the first of equations (9),

$$A = \frac{3y}{4} - \frac{p_4}{20}. \tag{14}$$

Therefore, $$(k^2 + t^2 y)^2 - y(2kt + y)^2 = y\left(\frac{3y}{4} - \frac{p_4}{20}\right)^2. \tag{15}$$

And, from (11),

$$8k(k^2 + t^2 y) = (16k^3 + p_5 y) - t\left(50y^2 - \frac{2}{5}p_4 y\right).$$

Substitute in (15) the value of $k^2 + t^2 y$ here given. The result is

$$t^2\left\{\left(50y^2 - \frac{2}{5}p_4 y\right)^2 - 256k^4 y\right\} + t\left\{-2(16k^3 + p_5 y)\left(50y^2 - \frac{2}{5}p_4 y\right) - 256k^3 y^2\right\}$$

$$= 64k^2 y\left(\frac{3y}{4} - \frac{p_4}{20}\right)^2 + 64k^2 y^3 - (16k^3 + p_5 y)^2. \tag{16}$$

§6. In (11) and (16) we have two equations, with the two unknown quantities y and t. The equations may be written

$$\left.\begin{aligned} 8kyt^2 + myt &= n \\ vyt^2 + qyt &= r \end{aligned}\right\} \qquad (17)$$

and

where
$$m = 50y - \frac{2}{5}p_4,$$
$$n = 8k^3 + p_5 y,$$
$$v = y\left(50y - \frac{2}{5}p_4\right)^2 - 256k^4$$
$$= 2500y^3 - 40p_4y^2 + \frac{4}{25}p_4^2y - 256k^4,$$
$$q = -2(16k^3 + p_5y)\left(50y - \frac{2}{5}p_4\right) - 256k^3y$$
$$= -100p_5y^2 + \left(\frac{4}{5}p_4p_5 - 1856k^3\right)y + \frac{64}{5}k^3p_4,$$
$$r = 100k^2y^3 - \left(\frac{24p_4k^2}{5} + p_5^2\right)y^2 + \left(\frac{4}{25}p_4^2k^2 - 32k^3p_5\right)y - 256k^6.$$

§7. The elimination of t from the equations (17) gives us

$$(vn - 8kr)^2 + y(qn - mr)(vm - 8kq) = 0. \qquad (18)$$

From the values of m, n, v, q and r in §6,

$$vn = 2500p_5y^4 + (20000k^3 - 40p_4p_5)y^3 + \left(\frac{4}{25}p_4^2p_5 - 320p_4k^3\right)y^2$$
$$+ \left(\frac{32}{25}p_4^2k^3 - 256k^4p_5\right)y - 2048k^7;$$
$$8kr = 800k^3y^3 - \left(\frac{192}{5}p_4k^3 + 8p_5^2k\right)y^2 + \left(\frac{32}{25}p_4^2k^3 - 256k^4p_5\right)y - 2048k^7;$$
$$qn = -100p_5^2y^3 + \left(\frac{4}{5}p_4p_5^2 - 2656k^3p_5\right)y^2 + \left(\frac{96}{5}p_5k^3p_4 - 14848k^6\right)y + 512k^6p_4;$$
$$mr = 5000k^2y^4 - (280p_4k^2 + 50p_5^2)y^3 + \left(\frac{248}{25}p_4^2k^2 - 1600k^3p_5 + \frac{2}{5}p_4p_5^2\right)y^2$$
$$- \left(12800k^6 + \frac{8}{125}p_4^3k^2 - \frac{64}{5}k^3p_4p_5\right)y + \frac{512}{5}k^6p_4;$$
$$vm = 125000y^4 - 3000p_4y^3 + 24p_4^2y^2 - \left(12800k^4 + \frac{8}{125}p_4^3\right)y + \frac{512}{5}p_4k^4;$$
$$8kq = -800p_5ky^2 + \left(\frac{32}{5}kp_4p_5 - 14848k^4\right)y + \frac{512}{5}p_4k^4.$$

$$\therefore vn - 8kr = y^2\left\{2500p_5y^2 + (19200k^3 - 40p_4p_5)y\right.$$
$$\left. + \left(\frac{4}{25}p_4^2p_5 - \frac{1048}{5}p_4k^3 + 8p_5^2k\right)\right\},$$

$$qn - mr = y\left\{-5000k^2y^3 + (-50p_5^2 + 280p_4k^2)y^2 + \left(\frac{2}{5}p_4p_5^2 - 1056k^3p_5 - \frac{248}{25}p_4^2k^2\right)y + k^2\left(-2048k^4 + \frac{8}{125}p_4^3 + \frac{32}{5}kp_4p_5\right)\right\};$$

$$vm - 8kq = y\left\{125000y^3 - 3000p_4y^2 + (24p_4^2 + 800kp_5)y - \left(-2048k^4 + \frac{8}{125}p_4^3 + \frac{32}{5}kp_4p_5\right)\right\}.$$

By substituting in (18) these values of $vn - 8kr$, $qn - mr$, $vm - 8kq$, we get

$$F(y) = q_0y^6 + q_1y^5 + \ldots + q_5y + q_6 = 0, \tag{19}$$

where

$q_0 = -625000000,$

$q_1 = 50000000p_4,$

$q_2 = -200000(200kp_5 + 11p_4^2),$

$q_3 = 102400000k^4 + 1280000kp_4p_5 + 44800p_4^3,$

$q_4 = -(25600kp_4^2p_5 + 4096000k^4p_4 + 640000p_5^2k^2 + 448p_4^4),$

$q_5 = 8192k^2p_4p_5^2 - 2048 \times 1856k^5p_5 + 64k^2 + \frac{49152}{5}k^4p_4^2 + \frac{6144}{25}kp_4^3p_5 + \frac{6784}{3125}p_4^5,$

$q_6 = -\left(\frac{8}{125}p_4^3 - 2048k^4 + \frac{32}{5}kp_4p_5\right)^2.$

§8. Assuming now that the coefficients p_3, p_4, p_5 are commensurable quantities, let the commensurable root y of equation (19) be found. Then a^2z is known. Then, from (11) and (16), t or $\frac{c}{a}$ is found.

§9. At this stage, as was indicated in §3, two courses are open to us. One is to proceed to find u_1^5, u_2^5, u_3^5, u_4^5 without troubling ourselves to inquire what a, c, e, θ, ϕ and h are separately. This, the natural and the shortest course, we will now follow. Since a^2z and $\frac{c}{a}$ are known, their product acz is known. And, by (14), A is known. Therefore $czhe(\theta^2 - \phi^2z)$, which, by the last of equations (7), is equal to $-aczA$, is known. Hence, by the first of equations (8), B is known. We might even more simply, acz being known, find B from the second of equations (9). The second of equations (8) gives us

$$y(B'\sqrt{z}) = 2c\sqrt{z}(k^2 - c^2z) + a^2zk(a\sqrt{z}) - 2kAa\sqrt{z}. \tag{20}$$

Now acz is known, and it is the same as $(a\sqrt{z})(c\sqrt{z})$; consequently, the signs

with which $c\sqrt{z}$ and $a\sqrt{z}$ must be taken relatively to one another are known. Therefore $B'\sqrt{z}$ is known. Therefore $B+B'\sqrt{z}$ is known. But, by the manner in which B, B', B'' and B''' were taken, keeping in view the values of B'' and B''' in (6),

$$u_1^5 = B + B'\sqrt{z} + \sqrt{(hz + h\sqrt{z})},$$

and

$$u_4^5 = B + B'\sqrt{z} - \sqrt{(hz + h\sqrt{z})}.$$

$$\therefore\ (u_1u_4)^5 = (B + B'\sqrt{z})^2 - (hz + h\sqrt{z}).$$

And, by (7), $u_1u_4 = a\sqrt{z}$. Therefore $hz + h\sqrt{z} = (B + B'\sqrt{z})^2 - (a\sqrt{z})^5$. This gives us

$$u_1^5 = B + B'\sqrt{z} + \sqrt{\{(B + B'\sqrt{z})^2 - (a\sqrt{z})^5\}},$$

$$u_4^5 = B + B'\sqrt{z} - \sqrt{\{(B + B'\sqrt{z})^2 - (a\sqrt{z})^5\}},$$

$$u_2^5 = B - B'\sqrt{z} + \sqrt{\{(B - B'\sqrt{z})^2 + (a\sqrt{z})^5\}},$$

$$u_3^5 = B - B'\sqrt{z} - \sqrt{\{(B - B'\sqrt{z})^2 + (a\sqrt{z})^5\}}.$$

Hence $u_1 + u_4 + u_2 + u_3$, the root of the given quintic, is known.

To find a, c, e, θ, ϕ and h separately.

§10. If we desire to obtain the values of a, c, e, θ, ϕ and h separately, we may first find a by means of a quadratic equation. By (12) and the third of equations (7),

$$he^2(\theta^2 + \phi^2 z) = (k^2 + t^2 y) - a(2kt + y)$$

and

$$he^2(\theta^2 - \phi^2 z) = -aeA.$$

Therefore,

$$2he^2\theta^2 = (k^2 + t^2y) - a(2kt + y) - aeA.$$

Also, by (13),

$$2hze^2\theta\phi = az(2kt + y) - (k^2 + t^2y).$$

Therefore,

$$\frac{\theta}{\phi z} = \frac{(k^2 + t^2y) - a(2kt + y) - aeA}{az(2kt + y) - (k^2 + t^2y)}. \tag{21}$$

But, by (9), $B''' = 0$. Therefore, from the last of equations (8),

$$\frac{\theta}{\phi z} = \frac{2e(k^2 - c^2z) + 2(k^2 + c^2z) - z(4kc + a^3z)}{z\{(4kc + a^3z) - 2(k^2 + c^2z)\}}$$

$$= \frac{2e(k^2 - t^2y) + 2(k^2 + t^2y) - az(4kt + y)}{az(4kt + y) - 2z(k^2 + t^2y)}. \tag{22}$$

From (21) and (22),

$$\frac{(k^2 + t^2y) - a(2kt + y) - aeA}{az(2kt + y) - (k^2 + t^2y)} = \frac{2e(k^2 - t^2y) + 2(k^2 + t^2y) - az(4kt + y)}{az(4kt + y) - 2z(k^2 + t^2y)},$$

$$\therefore\ \frac{(k^2 + t^2y) - a(2kt + y) - aeA}{a^2z(2kt + y) - a(k^2 + t^2y)} = \frac{2ae(k^2 - t^2y) + 2a(k^2 + t^2y) - a^2z(4kt + y)}{a(a^2z)(4kt + y) - 2a^2z(k^2 + t^2y)},$$

or,

$$\frac{(k^2 + t^2y) - a(2kt + y) - aeA}{y(2kt + y) - a(k^2 + t^2y)} = \frac{2ae(k^2 - t^2y) + 2a(k^2 + t^2y) - y(4kt + y)}{ay(4kt + y) - 2y(k^2 + t^2y)}.$$

Put
$$\beta = (k^2 + t^2y) - a(2kt + y),$$
$$\gamma = y(2kt + y) - a(k^2 + t^2y),$$
$$\delta = ay(4kt + y) - 2y(k^2 + t^2y),$$
$$\sigma = 2a(k^2 + t^2y) - y(4kt + y),$$
$$\tau = k^2 + t^2y.$$
Then
$$\frac{\beta - aeA}{\gamma} = \frac{2ae\tau + \sigma}{\delta},$$
$$\therefore ae(\delta A + 2\gamma\tau) = \beta\delta - \gamma\sigma. \tag{23}$$
But $\beta\delta = -a^2y(2kt + y)(4kt + y) + ay(k^2 + t^2y)(8kt + 3y) - 2y(k^2 + t^2y)^2$

and $\gamma\sigma = -\ y^2(2kt + y)(4kt + y) + ay(k^2 + t^2y)(8kt + 3y) - 2a^2(k^2 + t^2y)^2$,
$$\therefore \beta\delta - \gamma\sigma = (y - a^2)\{y(2kt + y)(4kt + y) - 2(k^2 + t^2y)^2\}.$$
And $y - a^2 = a^2z - a^2 = a^2(z - 1) = a^2e^2$. Therefore
$$\beta\delta - \gamma\sigma = a^2e^2\{y(2kt + y)(4kt + y) - 2(k^2 + t^2y)^2\}.$$
Therefore, from (23),
$$\delta A + 2\gamma\tau = ae\{y(2kt + y)(4kt + y) - 2(k^2 + t^2y)^2\}, \tag{24}$$
$$\therefore (\delta A + 2\gamma\tau)^2 = a^2e^2\{y(2kt + y)(4kt + y) - 2(k^2 + t^2y)^2\}^2$$
$$= (y - a^2)\{y(2kt + y)(4kt + y) - 2(k^2 + t^2y)^2\}^2. \tag{25}$$
Now k, y and t are known. And, by (14), A is known. Therefore (25) is a quadratic equation from which a can be found. The quadratic has its roots commensurable, and care must be taken in each case to select that one which satisfies all the conditions of the problem. When a has been found, since a^2z and $\frac{c}{a}$ are known, z and c are known; and, because $z = e^2 + 1$, the absolute value of e is known. The sign with which e is to be taken is determined by (24). Next, to find θ, ϕ and h, the third and fourth of equations (9) are $B'' = 1$, $B''' = 0$. Hence, taking the values of B'' and B''' in (8),
$$\left.\begin{aligned} a^2ze &= 2e(k^2 - c^2z)\theta - 2(k^2 + c^2z)(\theta + \phi z) + 2(\theta + \phi)(4kc + a^3z) \\ 0 &= 2e(k^2 - c^2z)\phi + 2(k^2 + c^2z)(\theta + \phi) - (\theta + \phi z)(4kc + a^3z) \end{aligned}\right\} \tag{26}$$
But e, z, a, c, k are known. Therefore, from the simultaneous equations (26), θ and ϕ are known. Therefore, from (7), h is known.

Another way of finding a, c, e, θ, ϕ and h.

§11. The values of a, c, e, θ, ϕ and h may be arrived at in another way. Let B and $B'\sqrt{z}$ be found as in §9. Then, because (see §9)

$$hz + h\sqrt{z} = (B + B'\sqrt{z})^2 - (a\sqrt{z})^5,$$

we have

$$hz = B^2 + (B'\sqrt{z})^2,$$

and

$$h\sqrt{z} = 2B(B'\sqrt{z}) - (a\sqrt{z})^5.$$

Here the quantities to which hz and $h\sqrt{z}$ are equated are known. Therefore h and z are known. When z is known, because a^2z and $\frac{c}{a}$ are known, a and c are known. Finally, θ and ϕ are obtained, as in §10, from the equations (26).

§12. *First Example.*—To exemplify the theory, let us take the equation

$$x^5 + 3x^2 + 2x - 1 = 0. \tag{27}$$

Because $k = -\frac{p_3}{20} = -\frac{3}{20}$, $p_4 = 2$, $p_5 = -1$, the equations (11) and (16) become

$$\left.\begin{aligned} &\frac{6t^2y}{5} + t\left(\frac{4y}{5} - 50y^2\right) = y + \frac{27}{1000}, \\ &t^2\left(\frac{125^2}{9}y^4 - \frac{500y^3}{9} + \frac{4y^2}{9} - \frac{9y}{100}\right) + t\left(\frac{625y^3}{9} + \frac{583}{180}y^2 - \frac{3y}{50}\right) \\ &= \frac{25y^3}{16} - \frac{38y^2}{45} - \frac{13y}{200} - \frac{81}{40000} \end{aligned}\right\} \tag{28}$$

The equation (19), obtained by eliminating t from the equations (28), is

$$F(y) = \left(\frac{25\times125^4}{36}\right)y^6 - \left(\frac{125^4}{9}\right)y^5 + \left(\frac{37\times125^3}{18}\right)y^4 - \left(\frac{1241\times125^2}{90}\right)y^3 + \left(\frac{3209\times125}{36}\right)y^2 - \left(\frac{5159}{136}\right)y + \frac{109^2}{22500} = 0,$$

and this has the commensurable root $\frac{1}{125}$. Therefore

$$y = a^2z = \frac{1}{125}.$$

Hence, from the two equations (28),

$$t = \frac{c}{a} = \frac{7}{4}.$$

In subsequent examples, when y and t have been found, we shall proceed at once to find B and $B'\sqrt{z}$, as in §9, without inquiring what a, c, e, θ, ϕ and h are separately. But we desire to illustrate, in one instance, the method of

obtaining the values of these quantities which was described in §10, and we will use the present instance for that purpose. To find a we take the equation (25),

$$(\delta A + 2\gamma\tau)^2 = (y - a^2)\{y(2kt + y)(4kt + y) - 2(k^2 + t^2 y)^2\}^2. \tag{29}$$

By (14),

$$A = \frac{3y}{4} - \frac{p_4}{20} = -\frac{47}{500}.$$

Also, from the values of k, y and t above given,

$$2kt + y = -\frac{517}{1000},$$

$$4kt + y = -\frac{521}{1000},$$

$$k^2 + t^2 y = \frac{47}{1000}.$$

Therefore

$$\delta = ay(4kt + y) - 2y(k^2 + t^2 y) = -\frac{521a + 47}{62500},$$

$$\gamma = y(2kt + y) - a(k^2 + t^2 y) = \frac{-47(125a + 11)}{125000},$$

$$\tau = k^2 - t^2 y = -\frac{1}{500}.$$

Hence (29) becomes

$$125(323a + 29)^2 = 36^2(1 - 125a^2). \tag{30}$$

One root of this equation is $-\frac{11}{125}$. But this root proves on examination to be inadmissible. We must therefore take the other root, which is $-\frac{9439}{25^2 \times 13^2}$.

Then, since $c = \frac{7a}{4}$, and $a^2 z = \frac{1}{125}$, we have

$$a = -\frac{9439}{25^2 \times 13^2} = -\frac{9439}{25 \times 4225},$$

$$c = -\frac{7 \times 9439}{422500},$$

$$z = \frac{5 \times 4225^2}{9439^2},$$

$$e = \frac{398}{9439}.$$

The sign of e is determined in the way pointed out in §10. By means of the values of e, z, a, c that have been obtained, we get, from the two equations (26),

$$\theta = \frac{125}{199}, \ \phi = -\frac{18 \times 9439}{199 \times 845} \therefore \theta^2 - \phi^2 z = -\frac{125}{199}.$$

Therefore, from (7), keeping in view that $A = -\frac{47}{500}$,

$$h = \frac{47}{8}\left(\frac{9439}{13\times 25\times 125}\right)^2.$$

Also, from the first two of equations (8), $B = \frac{39}{100}$, and $B' = \frac{91\times 9439}{500\times 4225}$

$$\therefore\ B'\sqrt{z} = \frac{91\sqrt{5}}{500},$$

and
$$hz + h\sqrt{z} = \frac{1}{625^2}\left\{\frac{47}{8}\,(21125 + 9439\sqrt{5})\right\}$$

Therefore,

$$\left.\begin{aligned}
u_1^5 &= \frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right) + \frac{1}{625}\sqrt{\left\{\frac{47}{8}\,(21125 + 9439\sqrt{5})\right\}}\\
u_4^5 &= \frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right) - \frac{1}{625}\sqrt{\left\{\frac{47}{8}\,(21125 + 9439\sqrt{5})\right\}}\\
u_2^5 &= \frac{13}{100}\left(3 - \frac{7\sqrt{5}}{5}\right) + \frac{1}{625}\sqrt{\left\{\frac{47}{8}\,(21125 - 9439\sqrt{5})\right\}}\\
u_3^5 &= \frac{13}{100}\left(3 - \frac{7\sqrt{5}}{5}\right) - \frac{1}{625}\sqrt{\left\{\frac{47}{8}\,(21125 - 9439\sqrt{5})\right\}}
\end{aligned}\right\} \tag{31}$$

Therefore the root of the given quintic is known.

§13. To verify this result, we have $9439\sqrt{5} = 21106.2456$. Therefore

$$\frac{1}{625}\sqrt{\left\{\frac{47}{8}\,(21225 + 9439\sqrt{5})\right\}} = .79696796,$$

$$\frac{1}{625}\sqrt{\left\{\frac{47}{8}\,(21225 - 9439\sqrt{5})\right\}} = .01679462.$$

Also,
$$\frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right) = .79696437$$

and
$$-\frac{13}{100}\left(3 - \frac{7\sqrt{5}}{5}\right) = .01696437.$$

Therefore
$$\begin{aligned}
u_1^5 &= 1.593932\ , & u_1 &= 1.09773\\
-u_4^5 &= \ .00000359, & -u_4 &= \ .08147\\
-u_2^5 &= \ .00016975, & -u_2 &= \ .17618\\
-u_3^5 &= \ .03375899, & -u_3 &= \ .50778.
\end{aligned}$$

Therefore $u_1 + u_2 + u_3 + u_4 = .3323$. Therefore

$$\begin{aligned}
x^5 &= .004052\\
3x^2 &= .331248\\
2x &= .66448\\
&\overline{\ .99978\ }
\end{aligned}$$

§14. If we wish to exhibit the root as in §9, we find from (8) that

$$B + B'\sqrt{z} = \frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right).$$

Therefore, since $a^5z^2\sqrt{z} = -\frac{\sqrt{5}}{625^2}$,

$$u_1^5 = \frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right) + \sqrt{\left\{\left(\frac{13}{100}\right)^2\left(3 + \frac{7\sqrt{5}}{5}\right)^2 + \frac{\sqrt{5}}{625^2}\right\}},$$

with corresponding expressions for u_2^5, u_3^5, u_4^5. As a matter of fact,

$$\left(\frac{13}{100}\right)^2\left(3 + \frac{7\sqrt{5}}{5}\right) + \frac{\sqrt{5}}{625^2} = \frac{1}{625^2}\left\{\frac{47}{8}(21125 + 9439\sqrt{5})\right\}.$$

§15. It is interesting to observe the application of the theory to the equation

$$x^5 - 3x^2 + 2x + 1 = 0, \tag{32}$$

whose roots, with the signs changed, are the same as the roots of the equation (27). By reference to §7, keeping in view that $k = -\frac{p_3}{20}$, it will be seen that, wherever an odd power of p_3 occurs as a factor in a term of any one of the coefficients of the equation $F(y) = 0$, an odd power of p_5 occurs as a factor of the same term. It follows that, by changing the signs of both p_3 and p_5 in the equation (27), in other words, by passing from the equation (27) to the equation (32), $F(y)$ remains unchanged. Therefore the commensurable root of this equation, which we have seen to be $\frac{1}{125}$, is the value of a^2z for the equation (32) as well as for the equation (27). To find t or $\frac{c}{a}$, the equations (17) give us

$$yt = \frac{vn - 8kr}{vm - 8kq}.$$

The values of $vn - 8kr$ and $vm - 8kq$ given in §7 show that, in passing from (27) to (32), $vn - kr$ simply changes its sign, while $vm - kq$ remains unaltered. Hence t or $\frac{c}{a}$ has the same absolute value for the equation (32) as for the equation (27), the signs, however, being different in the two cases. Consequently, for the equation (27), $\frac{c}{a} = -\frac{7}{4}$. Thus we get

$$a = -\frac{9439}{25 \times 4225}, \quad c = \frac{7 \times 9439}{422500}, \quad z = \frac{5 \times 4225^2}{9439^2}, \quad e = -\frac{398}{9439}.$$

Hence, by the first two of equations (8),

$$B + B'\sqrt{z} = -\frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right).$$

Therefore, for the equation (32), the values of u_1^5, u_4^5, u_2^5 and u_3^5 are

$$u_1^5 = -\frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right) - \sqrt{\left\{\frac{13^2}{100^2}\left(3 + \frac{7\sqrt{5}}{5}\right)^2 + \frac{\sqrt{5}}{625^2}\right\}},$$

$$u_4^5 = -\frac{13}{100}\left(3 + \frac{7\sqrt{5}}{5}\right) + \sqrt{\left\{\frac{13^2}{100^2}\left(3 + \frac{7\sqrt{5}}{5}\right)^2 + \frac{\sqrt{5}}{625^2}\right\}},$$

$$u_2^5 = -\frac{13}{100}\left(3 - \frac{7\sqrt{5}}{5}\right) - \sqrt{\left\{\frac{13^2}{100^2}\left(3 - \frac{7\sqrt{5}}{5}\right)^2 - \frac{\sqrt{5}}{625^2}\right\}},$$

$$u_3^5 = -\frac{13}{100}\left(3 - \frac{7\sqrt{5}}{5}\right) + \sqrt{\left\{\frac{13^2}{100^2}\left(3 - \frac{7\sqrt{5}}{5}\right)^2 - \frac{\sqrt{5}}{625^2}\right\}}.$$

p_2 not assumed to be zero.

§17. Let us now consider the more general case in which p_2 is not assumed to be zero. The method that has been illustrated above is still applicable, though the labor of operation, in dealing with particular instances, is increased. Putting, as before, y for a^2z, and t for $\frac{c}{a}$, we form equations corresponding to (11) and (16); from these we obtain the values of y and t; then we find B and $B'\sqrt{z}$ from equations corresponding to the first two of the group (8); or, B can be more readily found from the second of equations (6), and, on the principles of §9, when $B + B'\sqrt{z}$ is known, u_1u_4 or $g^2 - y$ being also known, the root of the given quintic is known.

§18. The values of B and $B'\sqrt{z}$ which correspond, when g or $-\frac{p_2}{10}$ is not zero, to those given in (8) for the case in which g is zero, are obtained from the equations (2) and (3) by keeping in view that, according to the first of equations (4), B is the rational part and B' the coefficient of $\sqrt{z}$ in the expansion of u_1^5.

Put

$$\left.\begin{aligned} A' &= eh(\theta^2 - \phi^2 z) \\ P &= 2k(k^2 - t^2y) - (g^2 - y)(gk - ty) \\ Q &= 2t(k^2 - t^2y) - (g^2 - y)(k - gt) \end{aligned}\right\} \qquad (33)$$

Then

$$\left.\begin{aligned} (g^2 - y)^2 B &= (g^2 + y)P + 2gyQ + 2azA'\{t(g^2 + y) + 2gk\} \\ \text{and} \quad (g^2 - y)^2 B' &= a\{2gP + (g^2 + y)Q\} + 2A'\{k(g^2 + y) + 2gty\} \end{aligned}\right\} \qquad (34)$$

By (5) and (6)

$$A(g^2-y)=g(k^2-t^2y)+azA',$$

and

$$p_4=-20A+5g^2+15y.$$

Therefore

$$azA'=\frac{g^2-y}{20}(5g^2+15y-p_4)-g(k^2-t^2y). \tag{35}$$

§19. From the second of equations (6),

$$p_5(g^2-y)^2+4B(g^2-y)^2-40ty(g^2-y)^2=0.$$

Therefore, from the first of the two equations (34),

$$(p_5-40ty)(g^2-y)^2+4(g^2+y)P+8gyQ+8azA'\{t(g^2+y)+2gk\}=0.$$

Putting for azA' its value in (35),

$$(g^2-y)\Big[(p_5-40ty)(g^2-y)+2\Big(g^2+3y-\frac{1}{5}p_4\Big)\{t(g^2+y)+2gk\}\Big]$$
$$+\big[4(g^2+y)P+8gazQ-8g(k^2-t^2y)\{t(g^2+y)+2gk\}\big]=0. \tag{36}$$

But, from the values of P and Q,

$$4(g^2+y)P+8gazQ-8g(k^2-t^2y)\{t(g^2+y)+2gk\}$$
$$=\{-4(g^2+y)(gk-ty)+8gy(k-gt)-8(k+gt)(k^2-t^2y)\}(g^2-y).$$

Therefore, rejecting the common factor g^2-y, (36) becomes

$$(p_5-40ty)(g^2-y)+2\Big(g^2+3y-\frac{1}{5}p_4\Big)\{t(g^2+y)+2gk\}$$
$$-4(g^2+y)(gk-ty)+8gy(k-gt)-8(k+gt)(k^2-t^2y)=0.$$

Arranging according to the powers of y,

$$\left.\begin{aligned}&50y^2t+y\Big\{8gt^3+8kt^2-t\Big(20g^2+\frac{2}{5}p_4\Big)-p_5\Big\}\\&+t\Big\{2g^2\Big(g^2-\frac{p_4}{5}\Big)-8gk^2\Big\}+g^2p_5-8k^3-\frac{4}{5}gkp_4=0.\end{aligned}\right\} \tag{37}$$

This is one equation between the unknown quantities y and t.

§20. From the last two of the equations (6),

$$he^2(\theta^2+\phi^2z)=(k^2+t^2y)-2kat+(g^2-y)(a-g)$$

and

$$2hze^2(\theta\phi)=2kzat-(k^2+y^2t)-(g^2-y)(az-g).$$

But

$$(\theta^2-\phi^2z)^2=(\theta^2+\phi^2z)^2-4z\theta^2\phi^2.$$

Therefore

$$zh^2e^4(\theta^2-\phi^2z)^2=z\{(k^2+t^2y)-2kat+(g^2-y)(a-g)^2\}$$
$$-\{2kzat-(k^2+t^2y)-(g^2-y)(az-g)^2\}.$$

Therefore

$$z\{he(\theta^2-\phi^2 z)\}^2=(k^2-t^2y)^2+(g^2-y)^3+(g^2-y)\{4kty-2g(k^2+t^2y)\}.$$

But $A'=he(\theta^2-\phi^2 z)$. Therefore, by (36),

$$\frac{1}{y}\left\{\frac{g^2-y}{20}(5g^2+15y-p_4)-g(k^2-t^2y)\right\}^2=(k^2-t^2y)^2+(g^2-y)^3$$
$$+(g^2-y)\{4kty-2g(k^2+t^2y)\};$$

or, arranging according to the powers of y,

$$\begin{aligned}&25y^3-y^2\left(16t^4+56gt^2-64kt+\frac{6}{5}p_4+35g^2\right)\\&-y\left\{8gt^2\left(g^2-\frac{1}{5}p_4\right)-32k^2t^2+\left(g^2-\frac{1}{5}p_4\right)\left(5g^2+\frac{1}{5}p_4\right)+8gk^2-16g^4\right\}\\&-g^2\left(g^2-\frac{1}{5}p_4\right)^2+8gk^2\left(g^2-\frac{1}{5}p_4\right)-16k^4=0.\end{aligned}\tag{38}$$

This is the second equation between the unknown quantities y and t.

§21. We may now either eliminate t from the two equations (37) and (38) so as to obtain an equation

$$F(y)=0$$

whose coefficients are rational functions of the coefficients of the quintic to be solved, or we may eliminate y so as to obtain an equation

$$\psi(t)=0$$

whose coefficients are rational functions of those of the quintic to be solved. In the former case, let the commensurable root y of the equation $F(y)=0$ be found. Then, by (37) and (38), t is known. In the latter case, let the commensurable root t of the equation $\psi(t)=0$ be found. Then, by (37) and (38), y is known. When y and t have thus been found, we find B and $B'\sqrt{z}$, exactly as in §12, from the equations (34), or B can more readily be found from the second of equations (6). Then

$$\begin{aligned}u_1^5&=B+B'\sqrt{z}+\sqrt{\{(B+B'\sqrt{z})^2-(u_1u_4)^5\}}\\&=B+B'\sqrt{z}+\sqrt{\{(B+B'\sqrt{z})^2-(g+a\sqrt{z})^5\}}.\end{aligned}$$

Therefore $\quad x=u_1+u_4+u_2+u_3$

$$\begin{aligned}&=[B+B'\sqrt{z}+\sqrt{\{(B+B'\sqrt{z})^2-(g+a\sqrt{z})^5\}}]^{\frac{1}{5}}\\&+[B+B'\sqrt{z}-\sqrt{\{(B+B'\sqrt{z})^2-(g+a\sqrt{z})^5\}}]^{\frac{1}{5}}\\&+[B-B'\sqrt{z}+\sqrt{\{(B-B'\sqrt{z})^2-(g-a\sqrt{z})^5\}}]^{\frac{1}{5}}\\&+[B-B'\sqrt{z}-\sqrt{\{(B-B'\sqrt{z})^2-(g-a\sqrt{z})^5\}}]^{\frac{1}{5}}.\end{aligned}$$

It need scarcely be pointed out that since $y=a^2z$, $a\sqrt{z}$ is known.

§22. *Second Example.*—As an illustrative example, let

$$x^5 - 10x^3 - 20x^2 - 1505x - 7412 = 0.$$

Here $g = -\frac{p_2}{10} = 1$, and $k = -\frac{p_3}{20} = 1$. Therefore the equations (37) and (38) become

$$50y^2t + y(8t^3 + 8t^2 + 528t + 7412) + 596t - 6216 = 0,$$

and

$$25y^3 - (16t^4 + 56t^2 - 64t - 1771)y^2 - (2384t^2 - 89384)y - 88804 = 0.$$

The commensurable values of y and t which satisfy these equations are $y = 2$, $t = -4$. Then, we can get the values of B and $B'\surd z$ from the equations (34), keeping in view that azA' is known by (35). As far as B is concerned, it is simpler to make use of the second of equations (6), keeping in view that $acz = ty = -8$. Therefore

$$4B = 7412 - 320,$$
$$\therefore\ B = 1773.$$

In order to obtain the value of $B'\surd z$, we must find P, Q and azA'. By (33) and (35),

$$P = -62 + 9 = -53,$$
$$Q = 248 + 5 = 253,$$

and

$$azA' = -77 + 31 = -46.$$

Therefore

$$\begin{aligned} B'\surd z &= 653a\surd z + 2A'\surd z \\ &= 653a\surd z - \frac{26(azA')a\surd z}{a^2z} \\ &= a\surd z(653 + 598) = 1251\surd 2, \end{aligned}$$
$$\therefore\ B + B'\surd z = 9(197 + 139\surd 2).$$

Hence, since $u_1u_4 = g + a\surd z = 1 + \surd 2$,

$$\begin{aligned} x = {}& 9^{\frac{1}{5}}\left[(197 + 139\surd 2) + \surd\{(197 + 139\surd 2)^2 - \frac{1}{81}(1 + \surd 2)^5\right]^{\frac{1}{5}} \\ &+ 9^{\frac{1}{5}}\left[(197 + 139\surd 2) - \surd\{(197 + 139\surd 2)^2 - \frac{1}{81}(1 + \surd 2)^5\right]^{\frac{1}{5}} \\ &+ 9^{\frac{1}{5}}\left[(197 - 139\surd 2) + \surd\{(197 - 139\surd 2)^2 - \frac{1}{81}(1 - \surd 2)^5\right]^{\frac{1}{5}} \\ &+ 9^{\frac{1}{5}}\left[(197 - 139\surd 2) - \surd\{(197 - 139\surd 2)^2 - \frac{1}{81}(1 - \surd 2)^5\right]^{\frac{1}{5}}. \end{aligned}$$

To verify this result,

$$\frac{1}{81}(1+\sqrt{2})^5 = 1.012496,$$

$$\frac{1}{81}(1-\sqrt{2})^5 = -.000150535,$$

$$197+139\sqrt{2} \;= 393.57568,$$

$$(197+139\sqrt{2})^2 = 154901.8,$$

$$197-139\sqrt{2} \;= .424315,$$

$$(197-139\sqrt{2})^2 = .1800434,$$

$$\sqrt{\left\{(197+139\sqrt{2})^2 - \frac{1}{81}(1+\sqrt{2})^5\right\}} = 393.57436,$$

$$\sqrt{\left\{(197-139\sqrt{2})^2 - \frac{1}{81}(1-\sqrt{2})^5\right\}} = \quad .180194.$$

Therefore

$$u_1 = 5.888,$$

$$u_4 = \;.412,$$

$$u_2 = 1.502,$$

$$u_3 = -.276,$$

$$\therefore\; x = 7.526.$$

MODIFICATION OF THE METHOD TO MEET SPECIAL CASES.

First special case: When p_2 and p_3 are both zero.

§23. When p_2 and p_3 are both zero, a modification of the general method is rendered necessary by the circumstance that the equations (11) and (16), from which y and t are to be found, are then virtually one, and so are insufficient to give us the values of y and t. In fact, they become

$$\left.\begin{aligned} &t\left(50y^2 - \frac{2}{5}p_4 y\right) - p_5 y = 0 \\ \text{and}\quad &\left\{t\left(50y^2 - \frac{2}{5}p_4 y\right) - p_5 y\right\}^2 = 0 \end{aligned}\right\} \tag{39}$$

§24. In an article which appeared in No. 2, Vol. VII of this *Journal*, the present writer showed that, when p_2 and p_3 are both zero, p_4 and p_5 have the forms

$$\left.\begin{aligned} p_4 &= \frac{5n^4(3-m)}{16+m^2} \\ p_5 &= \frac{n^5(22+m)}{16+m^2} \end{aligned}\right\} \tag{40}$$

These expressions for p_4 and p_5 furnish the criterion of solvability for the quintic

$$x^5 + p_4 x + p_5 = 0. \tag{41}$$

The root of the equation is

$$x = \theta^{\frac{1}{5}} + \alpha\theta^{\frac{2}{5}} + \lambda\alpha^2\theta^{\frac{3}{5}} - \lambda\alpha^3\theta^{\frac{4}{5}},$$

where λ is a root of the quartic equation

$$\lambda^4 - m\lambda^3 - 6\lambda^2 + m\lambda + 1 = 0,$$

and

$$\alpha = -\frac{\lambda^2 + 1}{n\lambda(\lambda - 1)},$$

and

$$\theta = -\frac{n^5\lambda(\lambda - 1)^2}{(16 + m^2)(\lambda + 1)(\lambda^2 + 1)}.$$

In the same issue of the *Journal* in which these results were established, Mr. J. C. Glashan of Ottawa, in "Notes on the Quintic," gave the relations between the coefficients of the solvable quintic

$$x^5 + p_2 x^3 + p_3 x^2 + p_4 x + p_5 = 0,$$

and, in his wider formulæ, the forms of p_4 and p_5 in (40) are included. They were subsequently announced by Mr. Emory McClintock, who had discovered them independently. It is to be regretted that Mr. Glashan has not made public the method by which his conclusions were reached.

§25. From our present position the criterion of solvability of the quintic (41) can be at once deduced, and the solution of the equation effected more readily than by the process employed in the article of the *Journal* just referred to. For, put

$$m = \frac{4A}{y} \text{ and } n = 2t;$$

then the first of the equations (9) may be written

$$p_4 = 5y(3 - m). \tag{42}$$

Also, by the second of equations (9),

$$p_5 = -4B + 40ty. \tag{43}$$

But, from the first of equations (8), $B = -t(y + 2A)$. Therefore

$$p_5 = 44ty + 8At = 2ty(22 + m). \tag{44}$$

And, by (15), in connection with (14),

$$t^4 y - y^2 = A^2 = \frac{m^2 y^2}{16},$$

$$\therefore\ y = \frac{16t^4}{16 + m^2} = \frac{n^4}{16 + m^2}.$$

Therefore also
$$2ty = \frac{2tn^4}{16+m^2} = \frac{n^5}{16+m^2}.$$

By the substitution of these values of y and $2ty$ in (42) and (43), the formulæ (40) are obtained, n in (40) being what we have called $2t$. To find now the root of the equation (41), eliminate m from the equations (40). The result is a sextic equation,
$$\psi(n) = 0.$$

When the coefficients of the quintic (41) are commensurable, the sextic $\psi(n) = 0$ has a commensurable root. Let this be found. Then n is known. Consequently, since $n = 2t$, t is known. Then y is known from (39). Then B is obtained from the second of equations (9), and $B'\surd z$ from (20). Also
$$u_1u_4 = g + a\surd z = a\surd z = \surd y.$$

Therefore u_1u_4 is known. Therefore, as in §9, the root of the given quintic is found.

§26. *Third Example.*—As an illustrative example, let
$$x^5 + \frac{625}{4}x + 3750 = 0.$$

Here the equations furnishing the criterion of solvability are
$$\frac{625}{4} = \frac{5n^4(3-m)}{16+m^2},$$
$$3750 = \frac{n^5(22+m)}{16+m^2}.$$

These are satisfied by the values $m = 2$, $n = 5$. Therefore
$$t = \frac{5}{2}.$$

Therefore, by (39),
$$y = \frac{125}{4}.$$

Therefore, by the second of equations (9),
$$B = -\frac{625}{4}.$$

And, by (20),
$$yB'\surd z = -2(c^2z)(c\surd z) = -2(t^2y)(t\surd y),$$
$$\therefore\ B'\surd z = -2t^3\surd y = -\frac{625}{8}\surd 5.$$
$$\therefore\ B + B'\surd z = -\frac{625}{4}\left(1 + \frac{\surd 5}{2}\right).$$

And $u_1u_4 = a\sqrt{z} = \frac{5\sqrt{5}}{2}$. Therefore

$$x = \left[-\frac{625}{4}\left(1+\frac{\sqrt{5}}{2}\right) + \sqrt{\left\{\left(\frac{625}{4}\right)^2\left(1+\frac{\sqrt{5}}{2}\right)^2 - \left(\frac{5\sqrt{5}}{2}\right)^5\right\}}\right]^{\frac{1}{5}}$$
$$+ \left[-\frac{625}{4}\left(1+\frac{\sqrt{5}}{2}\right) - \sqrt{\left\{\left(\frac{625}{4}\right)^2\left(1+\frac{\sqrt{5}}{2}\right)^2 - \left(\frac{5\sqrt{5}}{2}\right)^5\right\}}\right]^{\frac{1}{5}}$$
$$+ \left[-\frac{625}{4}\left(1-\frac{\sqrt{5}}{2}\right) + \sqrt{\left\{\left(\frac{625}{4}\right)^2\left(1-\frac{\sqrt{5}}{2}\right)^2 + \left(\frac{5\sqrt{5}}{2}\right)^5\right\}}\right]^{\frac{1}{5}}$$
$$+ \left[-\frac{625}{4}\left(1-\frac{\sqrt{5}}{2}\right) - \sqrt{\left\{\left(\frac{625}{4}\right)^2\left(1-\frac{\sqrt{5}}{2}\right)^2 + \left(\frac{5\sqrt{5}}{2}\right)^5\right\}}\right]^{\frac{1}{5}}.$$

Second special case: When $u_1u_4 = u_2u_3$.

§27. In the case in which $u_1u_4 = u_2u_3$, $a = 0$. Consequently, if c is distinct from zero, t or $\frac{c}{a}$ is infinite; while, if c is zero, t assumes the form $\frac{0}{0}$. As we cannot here proceed by finding t, the general method has to be modified.

§28. Because $a = 0$, $y = a^2z = 0$, and $ty = \left(\frac{c}{a}\right)(a^2z) = 0$. Also $t^2y = c^2z$. Equation (5) becomes

$$gA = k^2 - c^2z.$$

Therefore, from the first of equations (6),

$$k^2 - c^2z = \frac{g}{20}(5g^2 - p_4), \tag{45}$$

and, from the second of equations (6),

$$p_5 = -4B. \tag{46}$$

From (33) and (35),

$$\begin{aligned} P &= 2k(k^2 - c^2z) - (g^2 - a^2z)(gk - acz) \\ &= 2k(k^2 - c^2z) - g^3k, \end{aligned}$$

and

$$aQ = 2at(k^2 - c^2z) - (g^2 - a^2z)(ak - gc) = g^3c.$$

Therefore, from the first of equations (34), taken in connection with (46),

$$\frac{1}{4}g^2p_5 + k\{2(k^2 - c^2z) - g^3\} + 2czA' = 0. \tag{47}$$

But, A' having been put for $he(\theta^2 - \phi^2z)$, we have, from the value of $z\{he(\theta^2 - \phi^2z)\}$, obtained in §20,

$$z(A')^2 = (k^2 - c^2z)^2 + g^6 - 2g^3(k^2 + c^2z). \tag{48}$$

Therefore, from (47), by the elimination of A',

$$4c^2z = \frac{\left[\frac{1}{4}g^2p_5 + k\{2(k^2 - c^2z) - g^3\}\right]^2}{(k^2 - c^2z)^2 + g^6 - 2g^3(k^2 + c^2z)}. \tag{49}$$

But, by (45), c^2z is known in terms of the coefficients of the given quintic. Therefore the equation (49) gives a relation necessarily subsisting between the coefficients p_2, p_3, p_4 and p_5 of the solvable quintic in which $u_1u_4 = u_2u_3$, in order that u_1u_4 may be equal to u_2u_3. To find now the root of the quintic, c^2z is known by (45), and B is known by (46). To find $B'\sqrt{z}$, making use of the values of P and aQ and $(A'\sqrt{z})^2$ obtained above, we have, from the second of equations (34),

$$g^2(B'\sqrt{z}) = c\sqrt{z}\{2(k^2 - c^2z) + g^3\} + 2k(A'\sqrt{z}). \tag{50}$$

Now, by the first of equations (34), keeping in view the value of azA' in (35),

$$\begin{aligned} g^2B &= g^2\{2k(k^2 - c^2z) - g^3k\} + 2azA'\left(\frac{g^2c}{a}\right) \\ &= g^2\{2k(k^2 - c^2z) - g^3k\} + 2g^2czA'. \end{aligned}$$

As B and c^2z are known, this equation determines the sign of czA' or $(c\sqrt{z})(A'\sqrt{z})$, and therefore determines the signs with which $c\sqrt{z}$ and $A'\sqrt{z}$ are to be taken relatively to one another. Hence $z(A')^2$ being given by (48), $B'\sqrt{z}$ is given by (50). Therefore $B + B'\sqrt{z}$ is known. And $u_1u_4 = g$. Therefore, because

$$u_1^5 = B + B'\sqrt{z} + \sqrt{\{(B + B'\sqrt{z}) - g^5\}},$$

the root of the quintic is known.

§29. *Fourth Example.*—As an illustrative example, let

$$x^5 - \frac{22}{5}x^3 - \frac{11}{25}x^2 + \frac{11 \times 42}{125}x + \frac{11 \times 89}{3125} = 0.$$

Finding the value of c^2z from (45), and substituting in (49), we find that (49) is satisfied. Then, as in the preceding section,

$$p_5 = -4B \therefore B = -\frac{11 \times 89}{4 \times 5^5}.$$

The value of c^2z is $\frac{5}{16}\left(\frac{11}{25}\right)^2$. Therefore

$$k^2 - c^2z = \left(\frac{11}{25}\right)^2\left(\frac{1}{400} - \frac{5}{16}\right) = -\frac{31}{100}\left(\frac{11}{25}\right)^2,$$

and

$$k^2 + c^2z = \left(\frac{11}{25}\right)^2\left(\frac{1}{400} + \frac{5}{16}\right) = \frac{63}{200}\left(\frac{11}{25}\right)^2.$$

From (48), $$(A'\sqrt{z})^2 = \left(\frac{11^2 \times 5}{4 \times 5^5}\right)^2.$$

The value of czA', obtained from (47), is negative. Therefore $c\sqrt{z}$ and $A'\sqrt{z}$ must have different signs. Put

$$c\sqrt{z} = \frac{11}{25}\frac{\sqrt{5}}{4}$$

and

$$A'\sqrt{z} = -\frac{1}{5}\left(\frac{11}{25}\right)^2\frac{\sqrt{5}}{4}.$$

Then, from (50), dividing by g^2,

$$B'\sqrt{z} = -\frac{11\sqrt{5}}{4 \times 5^3}.$$

And $u_1 u_4 = g$. Therefore

$$u_1^5 = \frac{-11}{4 \times 5^5}(89 + 25\sqrt{5}) + \sqrt{\left\{\frac{121}{16 \times 25^5}(89 + 25\sqrt{5})^2 - \left(\frac{11}{25}\right)^5\right\}}.$$

This is, in a different form, the value of u_1^5 given by Lagrange.

§30. *Fifth Example.*—In the instance just considered, c is distinct from zero. It may be well to give an example in which c is zero, as the mode adopted above of determining the sign of $A'\sqrt{z}$ does not apply in that case. Let

$$x^5 + 20x^3 + 20x^2 + 30x + 10 = 0.$$

Here $g = -2$, and $k = -1$. Therefore (45) becomes

$$20(1 - c^2 z) = -2(20 - 30).$$

Therefore $c^2 z = 0$. This value of $c^2 z$ satisfies (49). Then, by (46), $B = -\frac{5}{2}$.

From 48, $$(A'\sqrt{z})^2 = 1 + 64 + 16 = 81 \therefore A'\sqrt{z} = 9.$$

Because c is zero, we cannot determine the sign of $A'\sqrt{z}$ relatively to that of $c\sqrt{z}$. But the root is the same, whatever sign be taken. Having found $A'\sqrt{z}$, we have, from (50), $4B'\sqrt{z} = -2 \times 9 \therefore B'\sqrt{z} = -\frac{9}{2}$. Therefore

$$B + B'\sqrt{z} = -\frac{5}{2} - \frac{9}{2} = -7, \quad B - B'\sqrt{z} = -\frac{5}{2} + \frac{9}{2} = 2.$$

And $u_1 u_4 = g = -2$. Therefore

$$\begin{aligned}
u_1^5 &= -7 + \sqrt{\{49 - (-2)^5\}} = 2, \\
u_4^5 &= -7 - \sqrt{\{49 - (-2)^5\}} = -16, \\
u_2^5 &= 2 + \sqrt{\{4 + 2^5\}} = 8, \\
u_3^5 &= 2 - \sqrt{\{4 + 2^5\}} = -4.
\end{aligned}$$

Therefore $x = 2^{\frac{1}{5}} - 2^{\frac{2}{5}} + 2^{\frac{3}{5}} - 2^{\frac{4}{5}}$.

§31. In the article of the *American Journal of Mathematics* (Vol. VI, page 103) referred to in the opening paragraph of this paper, the solvable irreducible quintic in which u_1u_4 is equal to u_2u_3 was discussed, and the roots of the equation were shown to be determinable in terms of the coefficients p_2, p_3, etc., even while these coefficients have no definite numerical values assigned to them, but remain symbolical. The solution that has now been given is much simpler than the former; equally with the former, it is applicable to equations with symbolical coefficients, the assumption being of course made that the coefficients are related as in (49); and it possesses the advantage of being part of a general theory.

ADDITIONAL EXAMPLES.

§32. *Sixth Example.*—Let

$$x^5 + 320x^2 - 1000x + 4288 = 0.$$

Here $g = 0$, $k = -16$. Because $g = 0$, we use the formulæ (11) and (16). The commensurable values of y and t which satisfy (11) and (16) are

$$y = 8, \quad t = 6.$$

Also, by (14),

$$A = 56.$$

Therefore, from the second of equations (9) and from (20),

$$B = -16 \times 37, \quad B'\sqrt{z} = -16 \times 20,$$
$$\therefore \; B + B'\sqrt{z} = -16\,(37 + 20\sqrt{2}).$$

And $u_1u_4 = -2\sqrt{2}$. Therefore

$$\begin{aligned} x &= u_1 + u_4 + u_2 + u_3 \\ &= [-16\,(37 + 20\sqrt{2}) + \sqrt{\{256\,(37 + 20\sqrt{2})^2 - (-2\sqrt{2})^5\}}]^{\frac{1}{5}} \\ &\quad + [-16\,(37 + 20\sqrt{2}) - \sqrt{\{256\,(37 + 20\sqrt{2})^2 - (-2\sqrt{2})^5\}}]^{\frac{1}{5}} \\ &\quad + [-16\,(37 - 20\sqrt{2}) + \sqrt{\{256\,(37 - 20\sqrt{2})^2 - (2\sqrt{2})^5\}}]^{\frac{1}{5}} \\ &\quad + [-16\,(37 - 20\sqrt{2}) - \sqrt{\{256\,(37 - 20\sqrt{2})^2 - (2\sqrt{2})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§33. *Seventh Example.*—Let

$$\left(\frac{x}{10}\right)^5 + 40\left(\frac{x}{10}\right)^2 - 69\left(\frac{x}{10}\right) + 108 = 0,$$

or,

$$x^5 + 40000x^2 - 690000x + 10800000 = 0.$$

Here $g = 0$, $k = -2000$. Because $g = 0$, we use the formulæ (11) and (16).

The commensurable values of y and t which satisfy (11) and (16) are

$$y = 2000, \quad t = 50.$$

Also, by (14), $$A = 36000.$$

Therefore, from the second of equations (9) and from (20),

$$B = -1700000, \quad B'\sqrt{z} = -400000\sqrt{5}.$$

And $u_1 u_4 = -20\sqrt{5}$. Therefore

$$\begin{aligned} x &= u_1 + u_4 + u_2 + u_3 \\ &= [-100000(17 + 4\sqrt{5}) + \sqrt{\{100000^2(17 + 4\sqrt{5})^2 + (20\sqrt{5})^5\}}]^{\frac{1}{5}} \\ &+ [-100000(17 + 4\sqrt{5}) - \sqrt{\{100000^2(17 + 4\sqrt{5})^2 + (20\sqrt{5})^5\}}]^{\frac{1}{5}} \\ &+ [-100000(17 - 4\sqrt{5}) + \sqrt{\{100000^2(17 - 4\sqrt{5})^2 - (20\sqrt{5})^5\}}]^{\frac{1}{5}} \\ &+ [-100000(17 - 4\sqrt{5}) + \sqrt{\{100000^2(17 - 4\sqrt{5})^2 - (20\sqrt{5})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§34. *Eighth Example.*—Let

$$x^5 - 20x^3 + 250x - 400 = 0.$$

Here $g = 2$, $k = 0$. Because g is distinct from zero, we use not the formulæ (11) and (16) as in the two preceding examples, but (37) and (38). The commensurable values of y and t which satisfy (37) and (38) are

$$y = 2, \quad t = -2.$$

Therefore, from the second of equations (6), $B = 60$. The value of azA' is obtained from (35). Thus, since $(A'\sqrt{z})(a\sqrt{z}) = azA'$, and since $a\sqrt{z} = \sqrt{y} = \sqrt{2}$, the value of $A'\sqrt{z}$ is known. Then, by the second of equations (34), $B'\sqrt{z} = 44\sqrt{2}$.

Therefore $$B + B'\sqrt{z} = 4(15 + 11\sqrt{2}).$$

And $u_1 u_4 = g + a\sqrt{z} = 2 + \sqrt{2}$. Therefore

$$\begin{aligned} x &= u_1 + u_4 + u_2 + u_3 \\ &= [4(15 + 11\sqrt{2}) + \sqrt{\{16(15 + 11\sqrt{2})^2 - (2 + \sqrt{2})^5\}}]^{\frac{1}{5}} \\ &+ [4(15 + 11\sqrt{2}) - \sqrt{\{16(15 + 11\sqrt{2})^2 - (2 + \sqrt{2})^5\}}]^{\frac{1}{5}} \\ &+ [4(15 - 11\sqrt{2}) + \sqrt{\{16(15 - 11\sqrt{2})^2 - (2 - \sqrt{2})^5\}}]^{\frac{1}{5}} \\ &+ [4(15 - 11\sqrt{2}) - \sqrt{\{16(15 - 11\sqrt{2})^2 - (2 - \sqrt{2})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§35. *Ninth Example.*—Let

$$x^5 - 5x^3 + \frac{85}{8}x - \frac{13}{2} = 0.$$

Here $g = \frac{1}{2}$, $k = 0$. Because g is distinct from zero, we use the formulæ (37)

and (38). The commensurable values of y and t which satisfy (37) and (38) are

$$y = \frac{1}{8},\ t = -1.$$

Therefore, from the second of equations (6), $B = \frac{3}{8}$. Finding $A'\sqrt{z}$ as in the immediately preceding example, we have, from the second of equations (34), $B'\sqrt{z} = \frac{3}{8}\sqrt{2}$. Therefore

$$B + B'\sqrt{z} = \frac{3}{8}(1 + \sqrt{2}).$$

And $u_1 u_4 = \frac{1}{4}(2 + \sqrt{2})$. Therefore

$$\begin{aligned} x &= u_1 + u_4 + u_2 + u_3 \\ &= \left[\frac{3}{8}(1+\sqrt{2}) + \sqrt{\left\{\frac{9}{64}(1+\sqrt{2})^2 - \frac{1}{4^5}(2+\sqrt{2})^5\right\}}\right]^{\frac{1}{5}} \\ &+ \left[\frac{3}{8}(1+\sqrt{2}) - \sqrt{\left\{\frac{9}{64}(1+\sqrt{2})^2 - \frac{1}{4^5}(2+\sqrt{2})^5\right\}}\right]^{\frac{1}{5}} \\ &+ \left[\frac{3}{8}(1-\sqrt{2}) + \sqrt{\left\{\frac{9}{64}(1-\sqrt{2})^2 - \frac{1}{4^5}(2-\sqrt{2})^5\right\}}\right]^{\frac{1}{5}} \\ &+ \left[\frac{3}{8}(1-\sqrt{2}) - \sqrt{\left\{\frac{9}{64}(1-\sqrt{2})^2 - \frac{1}{4^5}(2-\sqrt{2})^5\right\}}\right]^{\frac{1}{5}}. \end{aligned}$$

§36. *Tenth Example.*—Let

$$x^5 + \frac{20x}{17} + \frac{21}{17} = 0.$$

This and the next two examples are intended as additional illustrations of the method to be followed when p_2 and p_3 are both zero. The equations furnishing the criterion of solvability given in §24 are

$$\frac{20}{17} = \frac{5n^4(3 - m)}{16 + m^2}$$

and

$$\frac{21}{17} = \frac{n^5(22 + m)}{16 + m^2},$$

and they are satisfied by the commensurable values $m = -1$, $n = 1$. But, by §25, $n = 2t$. Therefore $t = \frac{1}{2}$. Putting $k = 0$ and $t = \frac{1}{2}$, equation (11)

becomes
$$\frac{1}{2}\left(50y - \frac{2}{5}p_4\right) = p_5.$$
$$\therefore\ y = \frac{1}{17}\ \therefore\ a\surd z = \surd y = \frac{\surd 17}{17}.$$
Therefore also $c\surd z = t(a\surd z) = \frac{\surd 17}{34}$. Therefore, by (20),
$$yB'\surd z = -2(c^2z)(c\surd z)\ \therefore\ B'\surd z = -\frac{\surd 17}{68}.$$
And, by the second of equations (9), $B = -\frac{1}{68}$. Therefore
$$B + B'\surd z = -\frac{1}{68}(1+\surd 17).$$
And $u_1u_4 = \frac{\surd 17}{17}$. Therefore
$$\begin{aligned} x = &\left[-\frac{1}{68}(1+\surd 17) + \sqrt{\left\{\left(\frac{1+\surd 17}{68}\right)^2 - \left(\frac{\surd 17}{17}\right)^5\right\}}\right]^{\frac{1}{5}} \\ &+ \left[-\frac{1}{68}(1+\surd 17) - \sqrt{\left\{\left(\frac{1+\surd 17}{68}\right)^2 - \left(\frac{\surd 17}{17}\right)^5\right\}}\right]^{\frac{1}{5}} \\ &+ \left[-\frac{1}{68}(1-\surd 17) + \sqrt{\left\{\left(\frac{1-\surd 17}{68}\right)^2 + \left(\frac{\surd 17}{17}\right)^5\right\}}\right]^{\frac{1}{5}} \\ &+ \left[-\frac{1}{68}(1-\surd 17) - \sqrt{\left\{\left(\frac{1-\surd 17}{68}\right)^2 + \left(\frac{\surd 17}{17}\right)^5\right\}}\right]^{\frac{1}{5}}. \end{aligned}$$

§37. *Eleventh Example.*—Let
$$x^5 - \frac{4x}{13} + \frac{29}{65} = 0.$$
The equations furnishing the criterion of solvability are
$$-\frac{4}{13} = \frac{5n^4(3-m)}{16+m^2},$$
$$\frac{29}{65} = \frac{n^5(22+m)}{16+m^2},$$
and they are satisfied by the values $m = 7$, $n = 1$. By §25, $n = 2t$. Therefore $t = \frac{1}{2}$. Therefore, from (11), $y = \frac{1}{65}$. Therefore
$$a\surd z = \frac{\surd 65}{65}\ \therefore\ c\surd z = t(a\surd z) = \frac{\surd 65}{130}.$$
Therefore, by (20), $yB'\surd z = -2(c^2z)(c\surd z)$. Therefore $B'\surd z = -\frac{\surd 65}{260}$. And, by the second of equations (9), $B = -\frac{9}{260}$. Therefore
$$B + B'\surd z = -\frac{9+\surd 65}{260}.$$

And $u_1u_4 = \frac{\sqrt{65}}{65}$. Therefore

$$x = \left[-\frac{9+\sqrt{65}}{260} + \sqrt{\left\{\left(\frac{9+\sqrt{65}}{260}\right)^2 - \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}$$
$$+ \left[-\frac{9+\sqrt{65}}{260} - \sqrt{\left\{\left(\frac{9+\sqrt{65}}{260}\right)^2 - \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}$$
$$+ \left[-\frac{9-\sqrt{65}}{260} + \sqrt{\left\{\left(\frac{9-\sqrt{65}}{260}\right)^2 + \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}$$
$$+ \left[-\frac{9-\sqrt{65}}{260} - \sqrt{\left\{\left(\frac{9-\sqrt{65}}{260}\right)^2 + \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}.$$

§38. *Twelfth Example.*—Let

$$x^5 + \frac{10x}{13} + \frac{3}{13} = 0.$$

The equations furnishing the criterion of solvability are

$$\frac{10}{13} = \frac{5n^4(3-m)}{16+m^2},$$
$$\frac{3}{13} = \frac{n^5(22+m)}{16+m^2},$$

and they are satisfied by the values $m = -7$, $n = 1$. By §25, $n = 2t$. Therefore $t = \frac{1}{2}$. Therefore, from (11), $y = \frac{1}{65}$. Therefore

$$c\sqrt{z} = t\sqrt{y} = \frac{\sqrt{65}}{130}.$$

And, by (20), $yB'\sqrt{z} = -2(c^2z)(c\sqrt{z})$. Therefore $B'\sqrt{z} = -\frac{\sqrt{65}}{260}$. And, by the second of equations (9), $B = \frac{1}{52}$. Therefore

$$B + B'\sqrt{z} = \frac{5-\sqrt{65}}{260}.$$

And $u_1u_4 = \frac{\sqrt{65}}{65}$. Therefore

$$x_1 = \left[\frac{5-\sqrt{65}}{260} + \sqrt{\left\{\left(\frac{5-\sqrt{65}}{260}\right)^2 - \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}$$
$$+ \left[\frac{5-\sqrt{65}}{260} - \sqrt{\left\{\left(\frac{5-\sqrt{65}}{260}\right)^2 - \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}$$
$$+ \left[\frac{5+\sqrt{65}}{260} + \sqrt{\left\{\left(\frac{5+\sqrt{65}}{260}\right)^2 + \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}$$
$$+ \left[\frac{5+\sqrt{65}}{260} - \sqrt{\left\{\left(\frac{5+\sqrt{65}}{260}\right)^2 + \left(\frac{\sqrt{65}}{65}\right)^5\right\}}\right]^{\frac{1}{5}}.$$

§39. *Thirteenth Example.*—Let

$$x^5 + 110(5x^3 + 60x^2 + 800x + 8320) = 0.$$

Here $g = -55$, $k = -330$, and the commensurable values of y and t which satisfy (37) and (38) are

$$y = 5 \times 11^2, \quad t = 10.$$

Therefore, from the second of equations (6), $B = -220 \times 765$. Finding $A'\sqrt{z}$ as in the second example, we have, from the second of equations (34), $B'\sqrt{z} = -220 \times 337\sqrt{5}$. Therefore

$$B + B'\sqrt{z} = -220(765 + 337\sqrt{5}).$$

And $u_1u_4 = -11(5 + \sqrt{5})$. Therefore

$$\begin{aligned} x = {} & [-220(765 + 337\sqrt{5}) + \sqrt{\{220^2(765 + 337\sqrt{5})^2 - (-55 - 11\sqrt{5})^5\}}]^{\frac{1}{5}} \\ & + [-220(765 + 337\sqrt{5}) - \sqrt{\{220^2(765 + 337\sqrt{5})^2 - (-55 - 11\sqrt{5})^5\}}]^{\frac{1}{5}} \\ & + [-220(765 - 337\sqrt{5}) + \sqrt{\{220^2(765 - 337\sqrt{5})^2 - (-55 + 11\sqrt{5})^5\}}]^{\frac{1}{5}} \\ & + [-220(765 - 337\sqrt{5}) - \sqrt{\{220^2(765 - 337\sqrt{5})^2 - (-55 + 11\sqrt{5})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§40. *Fourteenth Example.*—Let

$$x^5 - 20x^3 - 80x^2 - 150x - 656 = 0.$$

Here $g = 2$, $k = 4$, and the commensurable values of y and t which satisfy (37) and (38) are

$$y = 2, \quad t = 2.$$

Therefore, from the second of equations (6), $B = 204$. Finding $A'\sqrt{z}$ as in the immediately preceding example, we have, from the second of equations (34), $B'\sqrt{z} = 144\sqrt{2}$. Therefore

$$B = 12 \times 17, \text{ and } B'\sqrt{z} = 144\sqrt{2},$$
$$\therefore B + B'\sqrt{z} = 12(17 + 12\sqrt{2}).$$

And $u_1u_4 = 2 + \sqrt{2}$. Therefore

$$\begin{aligned} x_1 &= u_1 + u_4 + u_2 + u_3 \\ &= [12(17 + 12\sqrt{2}) + \sqrt{\{144(17 + 12\sqrt{2})^2 - (2 + \sqrt{2})^5\}}]^{\frac{1}{5}} \\ &\quad + [12(17 + 12\sqrt{2}) - \sqrt{\{144(17 + 12\sqrt{2})^2 - (2 + \sqrt{2})^5\}}]^{\frac{1}{5}} \\ &\quad + [12(17 - 12\sqrt{2}) + \sqrt{\{144(17 - 12\sqrt{2})^2 - (2 - \sqrt{2})^5\}}]^{\frac{1}{5}} \\ &\quad + [12(17 - 12\sqrt{2}) - \sqrt{\{144(17 - 12\sqrt{2})^2 - (2 - \sqrt{2})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§41. *Fifteenth Example.*—Let

$$x^5 - 40x^3 + 160x^2 + 1000x - 5888 = 0.$$

Here $g = 4$, $k = -8$, and the commensurable values of y and t which satisfy (37) and (38) are

$$y = 8, \quad t = -4.$$

Therefore, from the second of equations (6), $B = 1152$. Finding $A'\sqrt{z}$ as in the two preceding examples, we have, from the second of equations (34), $B'\sqrt{z} = 816\sqrt{2}$. Therefore

$$B + B'\sqrt{z} = 48\,(24 + 17\sqrt{2}).$$

And $u_1 u_4 = 4 + 2\sqrt{2}$. Therefore

$$\begin{aligned} x = {} & [48\,(24 + 17\sqrt{2}) + \sqrt{\{48^2\,(24 + 17\sqrt{2})^2 - (4 + 2\sqrt{2})^5\}}]^{\frac{1}{5}} \\ & + [48\,(24 + 17\sqrt{2}) - \sqrt{\{48^2\,(24 + 17\sqrt{2})^2 - (4 + 2\sqrt{2})^5\}}]^{\frac{1}{5}} \\ & + [48\,(24 - 17\sqrt{2}) + \sqrt{\{48^2\,(24 - 17\sqrt{2})^2 - (4 - 2\sqrt{2})^5\}}]^{\frac{1}{5}} \\ & + [48\,(24 - 17\sqrt{2}) - \sqrt{\{48^2\,(24 - 17\sqrt{2})^2 - (4 - 2\sqrt{2})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§42. *Sixteenth Example.*—Let

$$\left(\frac{x}{2}\right)^5 - 50\left(\frac{x}{2}\right)^3 - 600\left(\frac{x}{2}\right)^2 - 2000\left(\frac{x}{2}\right) - 11200 = 0,$$

or

$$x^5 - 200x^3 - 4800x^2 - 32000x - 3200 \times 112 = 0.$$

Here $g = 20$, $k = 240$, and the commensurable values of y and t which satisfy (37) and (38) are

$$y = 80, \quad t = 20.$$

Therefore, from the second of equations (6), $B = 640 \times 165$. Finding $A'\sqrt{z}$ as in the preceding examples of the same type, we have, from the second of equations (34), $B'\sqrt{z} = 640 \times 73\sqrt{5}$. Therefore

$$B + B'\sqrt{z} = 640\,(165 + 73\sqrt{5}).$$

And $u_1 u_4 = 4\,(5 + \sqrt{5})$. Therefore

$$\begin{aligned} x = {} & [640\,(165 + 73\sqrt{5}) + \sqrt{\{640^2\,(165 + 73\sqrt{5})^2 - (20 + 4\sqrt{5})^5\}}]^{\frac{1}{5}} \\ & + [640\,(165 + 73\sqrt{5}) - \sqrt{\{640^2\,(165 + 73\sqrt{5})^2 - (20 + 4\sqrt{5})^5\}}]^{\frac{1}{5}} \\ & + [640\,(165 - 73\sqrt{5}) + \sqrt{\{640^2\,(165 - 73\sqrt{5})^2 - (20 - 4\sqrt{5})^5\}}]^{\frac{1}{5}} \\ & + [640\,(165 - 73\sqrt{5}) - \sqrt{\{640^2\,(165 - 73\sqrt{5})^2 - (20 - 4\sqrt{5})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§43. *Seventeenth Example.*—Let

$$x^5 + 110\,(5x^3 + 20x^2 - 360x + 800) = 0.$$

Here $g = -55$, $k = -110$, and the commensurable values of y and t which satisfy (37) and (38) are

$$y = 5 \times 11^2, \quad t = -10.$$

Therefore, from the second of equations (6), $B = -11 \times 7500$. Finding $A'\sqrt{z}$ as in preceding examples of the same type, we have, from the second of equations (34), $B'\sqrt{z} = -11 \times 2700\sqrt{5}$. Therefore

$$B + B'\sqrt{z} = -3300(25 + 9\sqrt{5}).$$

And $u_1u_4 = -11(5 + \sqrt{5})$. Therefore

$$\begin{aligned} x = {} & [-3300(25 + 9\sqrt{5}) + \sqrt{\{3300^2(25 + 9\sqrt{5})^2 + 11^5(5 + \sqrt{5})^5\}}]^{\frac{1}{5}} \\ & + [-3300(25 + 9\sqrt{5}) - \sqrt{\{3300^2(25 + 9\sqrt{5})^2 + 11^5(5 + \sqrt{5})^5\}}]^{\frac{1}{5}} \\ & + [-3300(25 - 9\sqrt{5}) + \sqrt{\{3300^2(25 - 9\sqrt{5})^2 + 11^5(5 - \sqrt{5})^5\}}]^{\frac{1}{5}} \\ & + [-3300(25 - 9\sqrt{5}) - \sqrt{\{3300^2(25 - 9\sqrt{5})^2 + 11^5(5 - \sqrt{5})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§44. *Eighteenth Example.*—Let

$$x^5 - 20x^3 + 320x^2 + 540x + 638 = 0.$$

Here $g = 2$, $k = -16$, and the commensurable values of y and t that satisfy (37) and (38) are $\quad y = 8, \; t = -5.$

Therefore, from the second of equations (6), and the second of equations (34),

$$B = -12 \times 166, \; B'\sqrt{z} = -12 \times 117\sqrt{2},$$
$$\therefore B + B'\sqrt{z} = -12(166 + 117\sqrt{2}).$$

And $u_1u_4 = 2(1 + \sqrt{2})$. Therefore

$$\begin{aligned} x = {} & [-12(166 + 117\sqrt{2}) + \sqrt{\{144(166 + 117\sqrt{2})^2 - 32(1 + \sqrt{2})^5\}}]^{\frac{1}{5}} \\ & + [-12(166 + 117\sqrt{2}) - \sqrt{\{144(166 + 117\sqrt{2})^2 - 32(1 + \sqrt{2})^5\}}]^{\frac{1}{5}} \\ & + [-12(166 - 117\sqrt{2}) + \sqrt{\{144(166 - 117\sqrt{2})^2 - 32(1 - \sqrt{2})^5\}}]^{\frac{1}{5}} \\ & + [-12(166 - 117\sqrt{2}) - \sqrt{\{144(166 - 117\sqrt{2})^2 - 32(1 - \sqrt{2})^5\}}]^{\frac{1}{5}}. \end{aligned}$$

§45. *Nineteenth Example.*—Let

$$x^5 - 20x^3 - 160x^2 - 420x - 8928 = 0.$$

Here $g = 2$, $k = 8$, and the commensurable values of y and t which satisfy (37) and (38) are $\quad y = 72, \; t = -\frac{7}{3}.$

Therefore, from the second of equations (6), $B = 552$. Finding $A'\sqrt{z}$ as in preceding examples of the same type, we have, from the second of equations (34), $B'\sqrt{z} = -284\sqrt{2}$. Therefore

$$B + B'\sqrt{z} = 4(138 - 71\sqrt{2}).$$

And $u_1 u_4 = 2 - 6\sqrt{2}$. Therefore

$$\begin{aligned} x = &\left[4(138 - 71\sqrt{2}) + \sqrt{\{16(138 - 71\sqrt{2})^2 - (2 - 6\sqrt{2})^5\}}\right]^{\frac{1}{5}} \\ &+ \left[4(138 - 71\sqrt{2}) - \sqrt{\{16(138 - 71\sqrt{2})^2 - (2 - 6\sqrt{2})^5\}}\right]^{\frac{1}{5}} \\ &+ \left[4(138 + 71\sqrt{2}) + \sqrt{\{16(138 + 71\sqrt{2})^2 - (2 + 6\sqrt{2})^5\}}\right]^{\frac{1}{5}} \\ &+ \left[4(138 + 71\sqrt{2}) - \sqrt{\{16(138 + 71\sqrt{2})^2 - (2 + 6\sqrt{2})^5\}}\right]^{\frac{1}{5}}. \end{aligned}$$

§46. *Twentieth Example.*—Let

$$x^5 - 20x^3 + 170x + 208 = 0.$$

Here $g = 2$, $k = 0$, and the commensurable values of y and t which satisfy (37) and (38) are

$$y = 2, \quad t = 2.$$

Then in the usual way we get

$$B + B'\sqrt{z} = -12(1 - \sqrt{2}).$$

And $u_1 u_4 = 2 + \sqrt{2}$. Therefore

$$\begin{aligned} x = &\left[-12(1 - \sqrt{2}) + \sqrt{\{144(1 - \sqrt{2})^2 - (2 + \sqrt{2})^5\}}\right]^{\frac{1}{5}} \\ &+ \left[-12(1 - \sqrt{2}) - \sqrt{\{144(1 - \sqrt{2})^2 - (2 + \sqrt{2})^5\}}\right]^{\frac{1}{5}} \\ &+ \left[-12(1 + \sqrt{2}) + \sqrt{\{144(1 + \sqrt{2})^2 - (2 - \sqrt{2})^5\}}\right]^{\frac{1}{5}} \\ &+ \left[-12(1 + \sqrt{2}) - \sqrt{\{144(1 + \sqrt{2})^2 - (2 - \sqrt{2})^5\}}\right]^{\frac{1}{5}}. \end{aligned}$$